Dept. of Math. Univ. of Oslo
Pure Mathematics No. 23
ISSN 0806–2439 December 2006

LÉVY-COPULA-DRIVEN FINANCIAL PROCESSES

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Abstract. This paper proposes a general non-Gaussian Ornstein-Uhlenbeck model for a joint financial process based on marginal Lévy measures joined by a Lévy copula. Simulated processes then result from choices of marginal measures and Lévy copulas, with resulting statistics and inferences. Selected for analysis are the 3/2-stable and Gamma marginal Lévy measures, along with Clayton, Gumbel, and Complementary Gumbel Lévy versions of ordinary [probability] copulas, with the last two being here introduced. A relationship between the original coupled subordinated processes and the terminal dependency relationship between the simulated variables is observed and calibrated. Normal inverse Gaussian and tempered stable measures are also noted, as are additional Lévy copulas constructed from the Gumbel and Frank ordinary copulas, with some analysis and suggestion for using them in future research.

1. Introduction

A recent work of Fred Espen Benth with the author (Benth and Kettler 2006) investigated the relationship between electricity and gas prices by estimating marginal distributions and a theoretical copula joining them. That study simulated the model process, concluding with option prices for the spark spread, the difference of these two prices.

This paper proposes a general non-Gaussian Ornstein-Uhlenbeck subordinated model for a joint financial process. The model is founded not on process laws and corresponding marginal distributions with an ordinary [probability] copula, but rather on marginal Lévy measures joined by a Lévy copula. Simulated processes then result from choices for these measures and copula. Statistical analysis produces summary results, and a section on theory probes the relationship between an originating subordinator and the terminal relationship of the simulated variables.

The principal inferences and conclusions of this study are that the choice of Lévy copula is not material in differentiating the character or statistics of the price series, and that the terminal ordinary copula of the logarithmic price relatives is nearly the independent copula, regardless of choice of subordinating Lévy processes. These findings imply that financial processes modeled in this fashion are robust across functional forms and parameter settings. As well, the resulting logarithmic price relatives exhibit marked departure from normal distributions, an anticipated result, given the character of the marginal driving Lévy measures. The

Date: 14 December 2006.

2000 *Mathematics Subject Classification.* Primary: 91B24, 91B70. Secondary: 62M10, 62M20.

1991 *Journal of Economic Literature Subject Classification.* C51, G13.

Key words and phrases. Lévy copula, finance, stochastic processes, model construction, simulation, time series.

The author wishes to thank Fred Espen Benth and Frank Proske for valuable insights.

The R Foundation for Statistical Computing made available the statistical packages for this study (R Development Core Team 2005; Würtz et al. 2005; Genz, Bretz, and Hothorn (R port) 2005).

calibrations are interesting, as evidenced in various summary statistics such as the Anderson-Darling test for normality.

2. A general subordinated model

The paper is a report on research into the joint behavior of stock prices when they are defined in a geometric process with dependence on subordinated pure jump Ornstein-Uhlenbeck Lévy process. Within the subordinated process one joins marginal Lévy measures by a specified Lévy copula to produce stochastic variables then introduced into the geometric process. This structure of subordination is a Background Driving Lévy Process (BDLP) in the manner of Barndorff-Nielsen and Shephard. See, *e.g.*, (Barndorff-Nielsen and Shephard 2001, Section 1.1, pp. 167–169).

Here is the setup, beginning with the coupled Ornstein-Uhlenbeck process in the two dimensional case.

$$\begin{aligned} \mathrm{d}Y_t^1 &= -\lambda_1 Y_t^1 \,\mathrm{d}t + \mathrm{d}L_t^1,\ Y_0^1 = 0 \\ \mathrm{d}Y_t^2 &= -\lambda_2 Y_t^2 \,\mathrm{d}t + \mathrm{d}L_t^2,\ Y_0^2 = 0, \end{aligned} \tag{2.1}$$

where L_t^1 and L_t^2 are the subordinators. The variables (Y_t^1, Y_t^2) then enter the geometric equations as follows.

$$\begin{aligned} \mathrm{d}\log S_t^1 &= (\mu_1 + \beta_1 Y_t^1)\,\mathrm{d}t + \sqrt{Y_t^1}\,\mathrm{d}B_t^1,\ \log S_0^1 = 0 \\ \mathrm{d}\log S_t^2 &= (\mu_2 + \beta_2 Y_t^2)\,\mathrm{d}t + \sqrt{Y_t^2}\,\mathrm{d}B_t^2,\ \log S_0^2 = 0, \end{aligned} \tag{2.2}$$

where $B_t^1 \perp\!\!\!\perp B_t^2$ are Brownian motions.

The experimental design then calls for simulation of the joint Ornstein-Uhlenbeck process of Equations (2.1) with a Lévy copula, followed by simulation of the joint geometric process of Equations (2.2). The study begins by examining the relationship of the subordinators through a Lévy copula, by example, and continues through analysis of the simulated joint stock price series, with accompanying tables and charts.

3. Random selection from a Lévy copula

Recall that a Lévy copula is like an ordinary copula in that it is a function which retains all of the dependence information of a Lévy measure, while leaving all of the remaining information in the marginal Lévy measures. Let $\nu(\,\mathrm{d}x\,\mathrm{d}y)$ be such a bivariate Lévy measure. *Tail integrals* of this measure, which are the analogues of distribution functions, are defined as follows. First for the joint measure, in this study supported on $\overline{\mathbb{R}}_+^2$,

$$U(a,b) := \int_b^\infty \int_a^\infty \nu(\,\mathrm{d}x\,\mathrm{d}y)$$

and for the marginal measures,

$$U_1(a) := \int_0^\infty \int_a^\infty \nu(\,\mathrm{d}x\,\mathrm{d}y)$$

$$U_2(b) := \int_b^\infty \int_0^\infty \nu(\,\mathrm{d}x\,\mathrm{d}y)$$

The Lévy copula $C_L(u,v)$, then, defined on the same domain, is this.

$$C_L(u,v) := U\big(U_1^{-1}(u), U_2^{-1}(v)\big),$$

or equivalently,

$$C_L\big(U_1(a), U_2(b)\big) := U(a,b),$$

assuming all inverses are defined in the generalized sense.

For illustration consider a Clayton-Lévy copula subordinator model with common α-stable marginal Lévy measures. This is one of the six pairwise choices of copula and marginal measure for the later simulations. At the heart of selecting a jump pair is the choice of point in the copular domain. A presentation on this process appears here (Tankov 2003, Example 5.1, p. 20), and follows this plan. Let $C(u,v)$ be a Clayton-Lévy copula as such.

$$C(u,v) := \Big(u^{-\theta} + v^{-\theta}\Big)^{-\frac{1}{\theta}}\,,\ (u,v) \in [0,\infty)^2,\ \theta > 0$$

To simulate a joint α-stable subordinator on the chosen unit time interval one generates processes X_s and Y_s given the common marginal tail integral of

$$U(x) = x^{-\alpha}, \tag{3.1}$$

for which the inverse is

$$U^{-1}(y) = y^{-\frac{1}{\alpha}} \tag{3.2}$$

The α-stable subordinator has finite activity if $\alpha < 1$ because $|x|$ integrates the measure of the small jumps. In the simulations, however, the choice is $\alpha = 3/2$ to be more representative of what is observed in the financial markets. Specifically in the present context, applying the Fundamental Theorem of the Calculus to $U(x)$,

$$-\int_0^1 xU'(x) = \frac{\alpha}{1-\alpha} > 0 \tag{3.3}$$

Call Γ_i the i^{th} jump time of a Poisson process with intensity λ, and select a pair $(W_{i,1}, W_{i,2})$ of independent uniform variates on $[0,1]$. Then,

$$\begin{aligned} X_s^{(1)} &= \sum_{i=1}^{\infty} U^{-1}(\Gamma_i)\mathbb{1}_{\{[0,s]\}}(W_{i,1}) \\ X_s^{(2)} &= \sum_{i=1}^{\infty} U^{-1}\big(F^{-1}(W_{i,2}|\Gamma_i)\big)\mathbb{1}_{\{[0,s]\}}(W_{i,1}), \end{aligned} \tag{3.4}$$

Further, given the conditional distribution on the copula as

$$F(v|u) = \frac{\partial F_\theta(u,v)}{\partial u} = \left[1 + \left(\frac{u}{v}\right)^\theta\right]^{-\left(1+\frac{1}{\theta}\right)}, \tag{3.5}$$

it follows that

$$F^{-1}(W_{i,2}|u) = u \cdot \left(W_{i,2}^{-\frac{\theta}{1+\theta}} - 1\right)^{-\frac{1}{\theta}} \tag{3.6}$$

Equations (3.4) appear also in (Cont and Tankov 2004, Chapter 6, Section 5, p. 195).

The Clayton-Lévy copula is the only one of the copulas chosen for simulation which admits a closed-form expression for the inverse of the conditional copula distribution. The others require numeric inversion procedures for their conditional distributions.

For the present modeling purposes one wishes to simulate the BDLP by selecting jump times $\{\Gamma_i\}, 1 \leq i \leq N_\lambda(T)$, from a standard Poisson process over a revised time interval $[0, T]$, and then to calculate paired jumps $\left\{x_i^{(1)}, x_i^{(2)}\right\}$ at these times.

As the distribution of a waiting time $\Delta_i := \Gamma_i - \Gamma_{i-1}$, with $\Gamma_0 = 0$, conventionally, is

$$\Phi(\Delta_i) = 1 - \exp(-\lambda\Delta_i) = W_{i,1},$$

so

$$\Delta_i = \Phi^{-1}(W_{i,1})$$

One then constructs the $\{\Gamma_i\}$ iteratively as

$$\Gamma_i = \Gamma_{i-1} + \Delta_i,$$

continuing until determining $\Gamma_{N_\lambda(T)}$.

Next, with a view to the discrete simulation, construct an inventory of $N_\lambda(T)$ paired jumps in the manner of Equations (3.4), of which this is the i^{th}.

$$\begin{aligned} x_i^{(1)} &= U^{-1}(\Gamma_i) \\ x_i^{(2)} &= U^{-1}\big(F^{-1}(W_{i,2}|\Gamma_i)\big) \end{aligned} \tag{3.7}$$

These jumps shall appear in order of the $\{W_{i,1}\}$, as well, in harmony with Equations (3.4).

Remark. This diagram shows the sequence of calculations to produce the i^{th} pair of jump components $\left(x_i^{(1)}, x_i^{(2)}\right)$.

$$\begin{tikzcd} \Gamma_i = u \arrow[r, "F^{-1}(W_{i,2}|\cdot)"] \arrow[d, "U^{-1}"'] & v \arrow[d, "U^{-1}"'] \\ x_i^{(1)} & x_i^{(2)} \end{tikzcd}$$

4. Models

One may address models other than the Clayton-Lévy subordinator model with α-stable margins by allowing either other copulas or other margins, or both. Further, one may consider bidirectional copulas and margins, meaning those non-trivially supported on $\overline{\mathbb{R}}^n \setminus \{0\}$, with or without subordination. Among the marginal choices are the Gamma, normal inverse Gaussian (NIG), and the tempered stable processes (including as a limit the variance gamma,) and the bidirectional α-stable process, among others. Copula choices include the Gumbel-Lévy, herein defined, and Complementary Gumbel-Lévy, called complementary because its generator is the inverse of the Gumbel-Lévy generator.[1] The study now proceeds to examine some combinations of these *seriatim.*

For the α-stable and Gamma processes tail integrals and their inverses exist in closed form. For the α-stable processes one has Equations (3.1) and (3.2). For the Gamma processes one has these.

$$U^{\mathrm{G}}(x) = \mathrm{e}^{-\varrho x}, \quad \nu > 0 \tag{4.1}$$

for which the inverse is

$$\left(U^{\mathrm{G}}\right)^{-1}(y) = \max\left\{0, -\frac{1}{\varrho}\log\left(\frac{y}{\nu}\right)\right\} \tag{4.2}$$

See (Barndorff-Nielsen and Shephard 2001, Section 2.3.4, p. 175).

The Lévy measure $\nu^{\mathrm{NIG}}(x)$ on $\overline{\mathbb{R}} \setminus \{0\}$ of the NIG$(\alpha, \beta, \mu, \delta)$ process is this, with notation of (Barndorff-Nielsen 1998, p. 47, Equation 2.9). $K_1(\cdot)$ is the modified Bessel function of third order and index 1. As well, $\delta > 0$ and $0 \le |\beta| \le \alpha$.

$$\nu^{\mathrm{NIG}}(x) = \frac{\delta\alpha}{\pi\,|x|}K_1(\alpha\,|x|)\mathrm{e}^{\beta x} \tag{4.3}$$

The Lévy measure $\nu^{\mathrm{TS}}(x)$ on $\overline{\mathbb{R}} \setminus \{0\}$ of the tempered stable processes is this, with notation of (Cont and Tankov 2004, Chapter 4, Section 5, p. 119, Equation 4.26). As well, c_-, c_+, λ_-, and λ_+ are positive coefficients, and $\alpha > 0$.

$$\nu^{\mathrm{TS}}(x) = \frac{c_-}{|x|^{1+\alpha}}\mathrm{e}^{-\lambda_-|x|}\mathbb{1}_{\{x<0\}} + \frac{c_+}{x^{1+\alpha}}\mathrm{e}^{-\lambda_+ x}\mathbb{1}_{\{x>0\}} \tag{4.4}$$

The limiting case for $\alpha = 0$, and $c := c_- = c_+$ is the Lévy measure of the variance gamma process. See (Tankov 2006, p. 3, Equation 2.4).

The inverse tail integrals of the NIG and tempered stable processes are only known by numerical approximation. Though these processes are of interest to financial economists and mathematicians, these ideas are left for future study. For pertinent reading on the relationship between process probability and Lévy densities, including that of the Gamma distribution, see (Barndorff-Nielsen 2000).

Both the NIG and tempered stable processes have infinite activity, for the measures do not integrate $|x|$ near $\{0\}$, *cf.* Equation(3.3).

Following are the functional representations of the named Lévy copulas, including a bidirectional Clayton-Lévy version (Tankov 2006, p. 6, Equation 3.1), adapted from ordinary copulas

[1]Your author has chosen these names in honor of the late Professor Emil Julius Gumbel, founder of extreme value theory and Nazi antagonist. As there are many ways to chose Lévy copulas inspired by ordinary copulas, these are only two such choices.

of the same names. See (Cherubini, Luciano, and Vecchiato 2004, p. 124) for a presentation on ordinary copulas. Included for comparison are the Product-Lévy (Independent) and Fréchet-Lévy upper limit copula $C_\uparrow(u,v)$; no analogous Lévy version exists for the Fréchet-Lévy lower limit copula.

Clayton-Lévy:

$$C(u,v) = \left(u^{-\theta} + v^{-\theta}\right)^{-\frac{1}{\theta}},\ \theta > 0 \tag{4.5}$$

Clayton-Lévy, bidirectional:

$$C_B(u,v) = \left(|u|^{-\theta} + |v|^{-\theta}\right)^{-\frac{1}{\theta}} \left(\eta \mathbb{1}_{\{uv\geq 0\}} - (1-\eta)\mathbb{1}_{\{uv<0\}}\right),\ \theta > 0 \tag{4.6}$$

Gumbel-Lévy:

$$C_G(u,v) = \exp\left\{\left[\left(\log(u+1)\right)^{-\theta} + \left(\log(v+1)\right)^{-\theta}\right]^{-\frac{1}{\theta}}\right\} - 1,\ \theta > 0 \tag{4.7}$$

Complementary Gumbel-Lévy:

$$C_{\overline{G}}(u,v) = \left\{\log\left[\exp\left(u^{-\theta}\right) + \exp\left(v^{-\theta}\right) - 1\right]\right\}^{-\frac{1}{\theta}},\ \theta > 0 \tag{4.8}$$

Product-Lévy (Independent) for marginal Lévy measures $\nu_1, \nu_2 \in [0,\infty]$:

$$C_\perp(v,z) = \begin{cases} u & : (u,v) \in [0,\nu_1] \times [\nu_2] \\ v & : (u,v) \in [0,\nu_2] \times [\nu_1] \\ u+v & : (u,v) = (\nu_1,\nu_2) \\ 0 & : \text{elsewhere} \end{cases} \tag{4.9}$$

Fréchet-Lévy Upper:

$$C_\uparrow(u,v) = \min(u,v) \tag{4.10}$$

The following functions generate, respectively, the Clayton-Lévy, Gumbel-Lévy, and Complementary Gumbel-Lévy copulas in Archimedean analogy to their corresponding ordinary copulas. In each case $\phi : [0,\infty] \to [0,\infty]$.

$$\phi_C(x) := x^{-\theta}$$
$$\phi_G(x) := [\log(x+1)]^{-\theta}$$
$$\phi_{\overline{G}}(x) := \exp\left(x^{-\theta}\right) - 1$$

For a discussion of Lévy copula generation see (Kallsen and Tankov 2004, Section 6, pp. 21–23) and (Tankov 2003, Proposition 4.5, pp. 15–16). Note that $\phi_G(\cdot)$ and $\phi_{\overline{G}}(\cdot)$ are inverses of each other (after reparameterizing θ to $1/\theta$ in either formulation.)

Figures 1 and 2 display a Clayton-Lévy copula and its level curves; Figures 3 and 4 display a Gumbel-Lévy copula and its level curves; Figures 5 and 6 display a Complementary Gumbel-Lévy copula and its level curves. In each case $\theta = 1$.

Observe the vertical scale of these. $C(20,20) = 10.0000$ for the Clayton-Lévy copula; $C_G(20,20) = 3.5826$ for the Gumbel-Lévy copula; $C_{\overline{G}}(20,20) = 10.2439$ for the Complementary Gumbel-Lévy copula. Compare these values with $C(20,\infty) = C(\infty,20) = 20$, as for the other (and all) Lévy copulas.

Alternative generation of Lévy copulas comes from reference to an ordinary copula by way of a generator $\psi : [0,1] \to [0,\infty]$. Such procedures extend the possibilities for creating useful copulas in empirical research. For instance, one can begin with ordinary copulas such as the Gumbel and Frank, respectively.

$$\begin{aligned} C_G(u,v) &= \exp\left\{-\left[\left((-\log u)^\theta + (-\log v)^\theta\right)^{\frac{1}{\theta}}\right]\right\} \\ &\quad \theta \in [1,\infty), \text{with Product copula for } \theta = 1 \\ C_F(u,v) &= -\frac{1}{\theta}\log\left[1 + \frac{\big(\exp(-\theta u) - 1\big)\big(\exp(-\theta v) - 1\big)}{\exp(-\theta) - 1}\right] \\ &\quad \theta \in (-\infty,\infty) \setminus \{0\}, \text{with Product copula for } \theta = 0_\pm \end{aligned}$$

For a discussion of Lévy copula generation in this form also see (Kallsen and Tankov 2004, *loc. cit.*) and (Tankov 2003, *loc. cit.*). An example of such a generator, as proffered in (Tankov 2004, Theorem 5.1, pp. 167–169) is $\psi(x) = x/(1-x)$; another is $\psi(x) = -\log(1-x)$.

5. Simulation

The simulation proceeds in two phases, the first to develop the subordinated process, as displayed in Equations (2.1), the second to develop the geometric process, as displayed in Equations (2.2). Six models are selected, taking 3/2-stable subordinators or the Gamma subordinators, and coupling them by a Clayton-Lévy, Gumbel-Lévy, or Complementary Gumbel-Lévy copula, with chosen parameters. The calculations include charts in the Clayton-Lévy copula choice to illustrate the findings.

5.1. **The subordinated process.** The way is clear now to devise an algorithm for generating sequences of jumps joined by a Lévy copula. This algorithm generalizes *mutatis mutandis* to marginal processes other than the α-stable and to copulas other than the Clayton-Lévy, as this paper explores in the sequel.

Consider now that $U(\cdot)$ is the tail integral of an arbitrary Lévy measure.

(1) Select λ and T, then create a series of jump times $\{\Gamma_i\}, 1 \le i \le N_\lambda(T)$, by exponential delay. Note that if $\varepsilon := U^{-1}(T)$, then jumps smaller than ε, defined now as *small jumps*, in the $\left\{x_i^{(1)}\right\}$ series will be eliminated, for

$$x_1^{(1)} \ge x_2^{(1)} \ge \ldots \ge x_{N_\lambda(T)}^{(1)} \ge \varepsilon,$$

owing to the monotonicity of $U(\cdot)$.

(2) Calculate an inventory of incremental jump component pairs $\left\{x_i^{(1)}, x_i^{(2)}\right\}$.

(3) Calculate $\left(Y_t^1, Y_t^2\right)$ iteratively as the accumulation of these jumps following interjump exponential declines. Select the jumps for inclusion at time Γ_j on the order of the $\{W_{i,1}\}$, now indexing the BDLP by the jump times, as follows.

$$
\begin{aligned}
Y_j^1 &= \mathrm{e}^{-\lambda_1 \Delta_j} Y_{j-1}^1 + \sum_{i=1}^{N_\lambda(T)} x_i^{(1)} \mathbb{1}_{\{\Gamma_{j-1}/T < W_{i,1} \le \Gamma_j/T\}}, \ Y_0^1 = 0 \\
Y_j^2 &= \mathrm{e}^{-\lambda_2 \Delta_j} Y_{j-1}^1 + \sum_{i=1}^{N_\lambda(T)} x_i^{(2)} \mathbb{1}_{\{\Gamma_{j-1}/T < W_{i,1} \le \Gamma_j/T\}}, \ Y_0^2 = 0
\end{aligned} \tag{5.1}
$$

The first terms on the right of Equations (5.1) represent the interjump exponential declines of the Ornstein-Uhlenbeck process, whereas the second terms represent the accumulated jumps occurring between times Γ_{j-1} and Γ_j. The jumps are indicated (literally) for inclusion by the $\{W_{i,1}\}$, but actually occur when the next jump time Γ_j appears. By this means the subordinator remains stationary in that the expected size of the accumulated jumps at a jump time is proportional to the waiting time.

Remark. Jumps catalogued by this algorithm in the $\left\{x_i^{(1)}\right\}$ series are defined *large jumps*, to complement the small jumps. Observe that ε is such that

$$
U(\varepsilon) = U\big(U^{-1}(T)\big) = T \tag{5.2}
$$

Thus the Lévy measure of the large jumps, and therefore the intensity of the compound Poisson process they represent, is T, independent of $U(\cdot)$. The small jumps, and a method for including them in the study, is the subject of Section 5.4.

5.2. **Finite sample bias.** In selecting pairs of jumps, the first coordinate jump, computed as in the first of Equations (3.7), is limited to a lower bound of ε, as reported in Subsection 5.1. The second coordinate jump, computed as in the second of Equations (3.7), is not so limited. In consequence, a bias exists in jump selection leading to expected lower values in the second jump. The phenomenon is most pronounced for the Clayton-Lévy copula, so the correction proposed is only implemented in that case.

To counteract the observed bias the simulations also restrict the second coordinate jump to a lower bound of ε. This selection arrives in a direct manner by choosing the uniform random variable $W_{i,2}$ not on the interval $[0, 1]$, but rather on the interval $[0, \bar{r}_i]$, with $\bar{r}_i = F(T|\Gamma_i)$ chosen by the following reasoning. The revised requirement is that

$$
x_i^{(2)} = U^{-1}\left(F^{-1}(W_{i,2}|\Gamma_i)\right) \ge \varepsilon \tag{5.3}
$$

So

$$
U\left(x_i^{(2)}\right) = F^{-1}(W_{i,2}|\Gamma_i) \le U(\varepsilon) = T
$$

by Equation (5.2) and because $U(\cdot)$ is monotone decreasing. Therefore,

$$
F\left(U\left(x_i^{(2)}\right)|\Gamma_i\right) = W_{i,2} \le F(U(\varepsilon)|\Gamma_i) = F(T|\Gamma_i) =: \bar{r}, \tag{5.4}
$$

independent of $U(\cdot)$, as $F(\cdot|\Gamma_i)$ is monotone increasing.

An alternative plan would be to require

$$
\mathrm{E}\left[x_i^{(2)}\right] = x_i^{(1)}
$$

This scheme, while better in some ways, would make $\bar{r}_i$ dependent on $U(\cdot)$, as revealed by Equation (5.3).

5.3. **The geometric process.** Herein one simply takes the $\left\{Y_j^1, Y_j^2\right\}$ terms developed by simulating the subordinated Ornstein-Uhlenbeck process, inserting them into the discrete time version of the geometric process, *cf.* Equations (2.2), as so. This is implementation of Euler's Method (first order) on the deterministic part.

$$\begin{aligned} \log S_j^1 &= \log S_{j-1}^1 + (\mu_1 + \beta_1 Y_{j-1}^1)\Delta_j + \sqrt{Y_{j-1}^1}\widehat{B}_{\Delta_j}^1,\ \log S_0^1 = 0 \\ \log S_j^2 &= \log S_{j-1}^2 + (\mu_2 + \beta_2 Y_{j-1}^2)\Delta_j + \sqrt{Y_{j-1}^2}\widehat{B}_{\Delta_j}^2,\ \log S_0^2 = 0, \end{aligned} \tag{5.5}$$

where $\widehat{B}_{\Delta_j}^1 \perp\!\!\!\perp \widehat{B}_{\Delta_j}^2$ are Brownian motions. Exponentiating the $\left\{\log S_j^1, \log S_j^2\right\}$ series allows the recovery of the $\left\{S_j^1, S_j^2\right\}$ series.

5.4. **Amussen-Rosiński modification.** The processes articulated in Section 4 are necessarily approximate in that small jumps, those below the threshold of ε such as those computed in the α-stable and Gamma processes in Equations (3.2) and (4.2), are ignored. One can improve on this methodology by employing a method articulated by Amussen and Rosiński to approximate the small jumps by a Brownian motion. The primary reference is (Amussen and Rosiński 2001), with additional presentations in (Rosiński 2006; Prause 1999; Rosiński 1991).

The essence of the argument, with results incorporated in the simulations of this study, is that one can approximate the small jumps of a Lévy process of infinite measure frequently, but not always, by a Brownian motion with drift. Therein, the authors provide a necessary and sufficient condition that the *normal approximation*, as this capability is called, does not hold for any process with finite Lévy measure, such as the compound Poisson process, nor for the Gamma process, but does hold for the α-stable process for the entire admissible set $\{\alpha \mid 0 < \alpha \leq 2\}$. See Equation (5.8) below. For the NIG process see (Amussen and Rosiński 2001, Theorem 2.1 and Proposition 2.1, and Examples 2.1–2.5, pp. 484–486).

For the simulations using Gamma Lévy margins, ν is set to T so that $\varepsilon^G := \left(U^G\right)^{-1}(T) = 0$, reflecting the state of the Gamma process as having no small jumps. Insofar as $U^G(0) = \nu < \infty$, the Gamma process has finite variation, and thus is a compound Poisson process.

Figure 7 displays conditional copula distribution functions in the manner of Equation (3.5), which appears for the Clayton-Lévy copula along with similar formulations for the Gumbel-Lévy and Complementary Gumbel-Lévy copulas. In each case the point of conditional evaluation is $u = 2$. The rank of vertical scaling described for the copulas is evident in these measures also for evaluations at $(u, v) = (2, 5)$, at the right hand boundary of this chart.

Figure 8 displays the marginal Lévy measure for the 3/2-stable subordinate process with parameter $\theta = 1$. For the Gamma subordinate process (not shown) the parameter choices are $\nu = T = 20$ and $\varrho = 1$. The formulations to determine drift $a(\varepsilon)$ and variance $s^2(\varepsilon)$ of the relevant Brownian motion for the α-stable process, adapted to the present circumstances, follow.

$$a(\varepsilon) = -\int_{\varepsilon}^{1} x\,\nu(\mathrm{d}x) \tag{5.6}$$

$$s^2(\varepsilon) = \int_0^{\varepsilon} x^2\,\nu(\mathrm{d}x) \tag{5.7}$$

The specific condition for the normal approximation to apply, with the α-stable process conforming, is this for Lévy measures without atoms in $(0, \epsilon)$.

$$\lim_{\varepsilon \to 0} \frac{s(\varepsilon)}{\varepsilon} = \infty$$

Substituting $-U'(x)\,\mathrm{d}x$ for $\nu(\mathrm{d}x)$ by the Fundamental Theorem of the Calculus, and recalling that $\varepsilon = U^{-1}(T)$, one has that

$$a(\varepsilon) = \tfrac{\alpha}{1-\alpha}\left(1 - \varepsilon^{1-\alpha}\right) = \tfrac{\alpha}{1-\alpha}\left(1 - T^{-\frac{1-\alpha}{\alpha}}\right) > 0, \quad 0 < \alpha \leq 2 \tag{5.8}$$

$$s^2(\varepsilon) = \tfrac{\alpha}{2-\alpha}\left(\varepsilon^{2-\alpha}\right) \qquad = \tfrac{\alpha}{2-\alpha}\left(T^{-\frac{2-\alpha}{\alpha}}\right) > 0, \qquad 0 < \alpha < 2 \tag{5.9}$$

Simple calculations show that

$$\begin{aligned} a(\varepsilon)|_{\alpha \to 1} &= \log T \\ a(\varepsilon)|_{\alpha = 2} &= -2\left(1 - T^{1/2}\right) \\ s^2(\varepsilon)|_{\alpha \to 2} &= \infty \end{aligned}$$

Note that the cases, $\alpha = 1$ and $\alpha = 2$ are the Cauchy and normal processes, respectively. As well, observe that by the assumption $T = 20$ the threshold for small jumps $\varepsilon = 0.1357$. In the $3/2$-stable process the marginal Lévy measure $\nu(\varepsilon) = 221.04$.

To include the normal approximation to the original formulation of the subordinated process is straightforward. Simply amend Equations (2.1) as follows.

$$\begin{aligned} \mathrm{d}Y_t^1 &= \left(-\lambda_1 Y_t^1 + \underline{a^{(1)}(\varepsilon)}\right)\,\mathrm{d}t + \underline{s^{(1)}(\varepsilon)\,\mathrm{d}\widehat{B}_t^1} + \mathrm{d}L_t^1,\ Y_0^1 = 0 \\ \mathrm{d}Y_t^2 &= \left(-\lambda_2 Y_t^2 + \underline{a^{(2)}(\varepsilon)}\right)\,\mathrm{d}t + \underline{s^{(2)}(\varepsilon)\,\mathrm{d}\widehat{B}_t^2} + \mathrm{d}L_t^2,\ Y_0^2 = 0, \end{aligned} \tag{5.10}$$

where the new terms are underlined, and where $\widehat{B}_t^1 \perp\!\!\!\perp \widehat{B}_t^2$ are Brownian motions.

These inclusions also translate to the realm of the simulation by modification to Equations (5.1), as here.

$$\begin{aligned} Y_j^1 &= \mathrm{e}^{-\lambda_1 \Delta_j} Y_{j-1}^1 + \underline{a^{(1)}(\varepsilon)\Delta_j} + \underline{s^{(1)}(\varepsilon)\widehat{B}_{\Delta_j}^1} + \sum_{i=1}^{N_\lambda(T)} x_i^{(1)} \mathbb{1}_{\{\Gamma_{j-1}/T < W_{i,1} \leq \Gamma_j/T\}},\ Y_0^1 = 0 \\ Y_j^2 &= \mathrm{e}^{-\lambda_2 \Delta_j} Y_{j-1}^1 + \underline{a^{(2)}(\varepsilon)\Delta_j} + \underline{s^{(2)}(\varepsilon)\widehat{B}_{\Delta_j}^2} + \sum_{i=1}^{N_\lambda(T)} x_i^{(2)} \mathbb{1}_{\{\Gamma_{j-1}/T < W_{i,1} \leq \Gamma_j/T\}},\ Y_0^2 = 0, \end{aligned} \tag{5.11}$$

where the new terms are underlined, and where $\widehat{B}_{\Delta_j}^1 \perp\!\!\!\perp \widehat{B}_{\Delta_j}^2$ are Brownian motions.

Observe that the sum of the underlined terms in either of Equations (5.11) could be negative with the random choice of a sufficiently negative Brownian path over the interval Δ_j. In these instances the sum of such terms is forced to zero to preserve the non-negative incremental characteristic of a subordinator. By the Doob Martingale Inequality the probability of these instances decreases exponentially with time, and therefore becomes insignificant.

5.5. **Stability.** At some level of the simulated subordinated Ornstein-Uhlenbeck process is in equilibrium. In Equation (5.1) this is where either Y_j^1 or Y_j^2 is such that the expected infinitesimal decline from the exponential term is matched by the expected infinitesimal advance from the pure jump term. Letting $k \in \{1, 2\}$, then at time Γ_j these respective rates are $\lambda_k Y_j^k$

and $\nu(\varepsilon,\infty) = U^{-1}(\varepsilon) = T = \lambda$ for a generic $U(\cdot)$. Thus for $Y_j^k = \lambda/\lambda_k$ the process has conditional expectation of Y_j^k, and therefore is a local martingale. It is desirable, consequently, to start each process at $Y_0^k = \lambda/\lambda_k$ to ensure stability from the onset.

In the related process with the Amussen-Rosiński modification as in Equation (5.11) the corresponding starting point is $Y_0^k = \lambda/\left(\lambda_k - a^{(k)}(\varepsilon)\right)$ to allow for the compensating drift of the Brownian approximation.

Accordingly, Equations (5.1) and (5.11) are reset to these starting values.

5.6. **Statistics.** The study examined six models. For margins the choice was either a 3/2-stable process, identical in each variable, or a Gamma process, also identical in each variable. For copulas the choice was either Clayton-Lévy, Gumbel-Lévy, or Complementary Gumbel-Lévy. Chosen parameters for the Gamma process were and $\nu = T = 20$, as noted, and $\varrho = 1$. In each copula $\theta = 1$. In the geometric processes $\mu = 0.001$ and $\beta = 0.10$ in each variable. By these choices the models were symmetric in all aspects, except for small residual biases in the choices of jump pairs owing to finite sample biases.

Statistics and tabular results are reported across the three copular models and the two marginal measures. Pseudo-random numbers used to generate sequences were the same for both the 3/2-stable and Gamma processes so that the generated paths are directly comparable.

Accompanying the text is a pair of charts for the 3/2-stable marginal choice for the Clayton-Lévy copula, illustrating the empirical copula which resulted from simulations of 500 paths. Additionally appear four pairs of charts illustrating features of a single random path from these selections.

Charts for the other copulas and for the Gamma margin are not shown in the interest of economy, as those charts are qualitatively similar to the ones which appear. A conclusion of this study is that the results of simulation are robust over the various choices, an idea to be revisited.

The first two charts of these 10 for the Clayton-Lévy copula show results of computing the empirical ordinary copula at the terminal prices. Figures 9 and 10 exhibit these respective axial views. Note that a copula for prices is the same as a copula for logarithmic prices, because the logarithm is an increasing function. See (Schweizer and Wolff 1981, Theorem 3, p. 881).

An exercise in fitting a Clayton ordinary copula to the empirical copula in each model gave the results appearing in Table 1. The model took $C_\gamma(u,v)$ as the empirical copula, with

$$C_\theta = (u^{-\theta} + u^{-\theta} - 1)^{-\frac{1}{\theta}},$$

the ordinary Clayton copula, and evaluated

$$\min_\theta \sum_{u,v} (C_\gamma - C_\theta)^2$$

Note this is the same as

$$\min_\theta \sum_{u,v} \left[\left(C_\gamma(u,v) - C_\perp(u,v)\right) - \left(C_\theta(u,v) - C_\perp(u,v)\right)\right]^2 \tag{5.12}$$

wherein the interpretation is that of comparing the differences of copulas to the independent copula.

The coefficient of determination r^2 was calculated by comparing the variances of the respective differences of the empirical copula and the Clayton copula as fit, to the independent

Copula	Margin	3/2-stable	Gamma
Clayton	θ	0.0703	0.0835
	objective	0.0121	0.0168
	r^2	0.7235	0.7711
Gumbel	θ	0.0197	0.0102
	objective	0.0162	0.0219
	r^2	0.6908	0.7939
Comple-mentary Gumbel	θ	0.0000	0.0000
	objective	0.0951	0.0914
	r^2	0.9176	0.9084

TABLE 1. Statistics fitting empirical copula

	Margin	3/2-stable		Gamma	
Copula	Variable	First	Second	First	Second
Clayton	A	1.8801	1.6225	1.4643	1.8964
	P	0.0001	0.0003	0.0008	0.0001
	Skewness	-1.4218	0.5737	-1.2053	0.7135
	Kurtosis	4.6838	0.6699	3.7792	1.6030
Gumbel	A	1.7602	2.2113	1.7031	2.2424
	P	0.0002	0.0000	0.0002	0.0000
	Skewness	-0.1250	0.9393	-0.0310	0.7047
	Kurtosis	2.6468	4.2463	3.5151	3.7896
Comple-mentary Gumbel	A	6.0889	1.2741	6.1703	1.0986
	P	0.0000	0.0025	0.0000	0.0067
	Skewness	4.5044	0.2667	4.1547	0.2061
	Kurtosis	30.4387	1.4195	26.3826	1.3461

TABLE 2. Anderson-Darling statistics of the logarithmic price relative series for the sample paths

copula $C_\perp(u,v) = uv$, for each of the models, as here.

$$r^2 = 1 - \mathrm{var}\big(C_\theta(u,v) - C_\perp(u,v)\big)/\mathrm{var}\big(C_\gamma(u,v) - C_\perp(u,v)\big)$$

This result follows the formulation of Equation (5.12).

Other methods to fit, including by maximum likelihood, are described here (Frees and Valdez 1998, Sec. 4, pp. 12–18).

Figures 11 and 12 show histograms of the logarithmic price relative series for the sample paths for the Clayton-Lévy copula with 3/2-stable margins. Anderson-Darling tests for normality cause rejection of the null hypothesis in each instance, as is evident from the histograms. Some statistics for the three copulas, and for both the 3/2-stable and Gamma margins, appear in Table 2.

Figures 13 and 14 show Q-Q probability plots of the logarithmic price relative series for the sample paths for the Clayton-Lévy copula with 3/2-stable margins. Figures 15 and 16 show subordinating pure jump processes for the sample paths for this combination. Figures 17 and 18 show prices for the sample paths, again for the same combination.

5.7. **Inferences.** Three principal inferences are discernible from the course of this study.

(1) The terminal logarithmic price relative empirical copulas are immaterially different from the independent copula, over all models. This fact is apparent from the entries of Table 1. The Clayton ordinary copula does provide a good fit, but the optimized parameter θ is close to zero in all cases (being a flat zero for the Complementary Gumbel-Lévy copula,) the independent limit of the Clayton family. Further, the projected views of the empirical copulas, as appearing in Figures 9 and 10 for the Clayton-Lévy copula with 3/2-stable margins show only patterns which are attributable to the accumulation of computational errors; specifically they exhibit low amplitude wave patterns typical of truncation errors in evaluating transcendental functions by series methods.

(2) The logarithmic price relative series are distinctly not normal, exhibiting significant skewness and kurtosis, as revealed by all the Anderson-Darling and related statistics appearing in Table 2. This is an expected result, given the nature of the driving 3/2-stable and Gamma Lévy marginal subordinating processes.

(3) The choice of copula is not important in determining the quality of the inferences in the two items above.

6. Conclusions

This study established that the proposed model provides a computationally reasonable scheme for generating financial processes. The model incorporates the freedom to describe the dependency relationship between variables with the generality of a Lévy copula, while also permitting flexible jump processes as often required.

Financial process modeling of the fashion proposed by this study appears to be robust across choices of marginal Lévy measures and Lévy copulas. Subtle distinctions are evident, but in general all of the developed processes are remarkably similar.

Planned future research includes delving into the theory of Lévy-copula-driven financial processes by establishing a set of first principles, thus enabling informed prediction of terminal processes and copulas from the subordinators, *ex ante*.

List of Figures

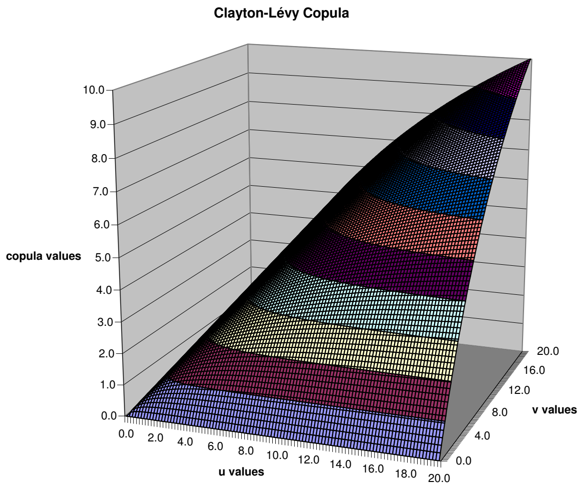

Figure 1. Clayton-Lévy Copula

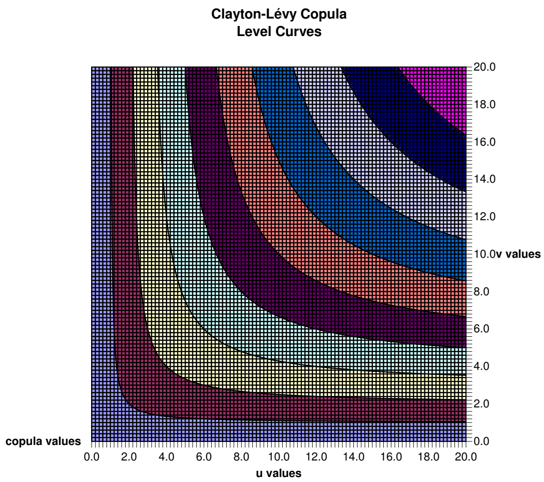

Figure 2. Clayton-Lévy Copula, Level Curves

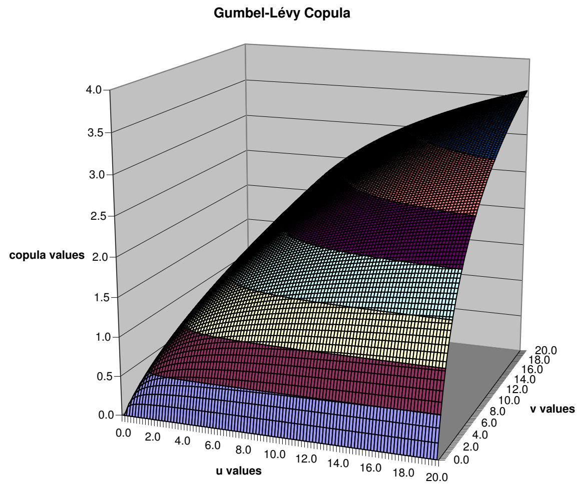

Figure 3. Gumbel-Lévy Copula

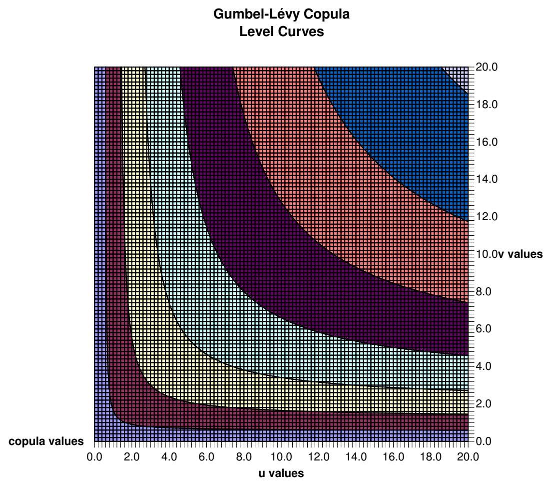

Figure 4. Gumbel-Lévy Copula, Level Curves

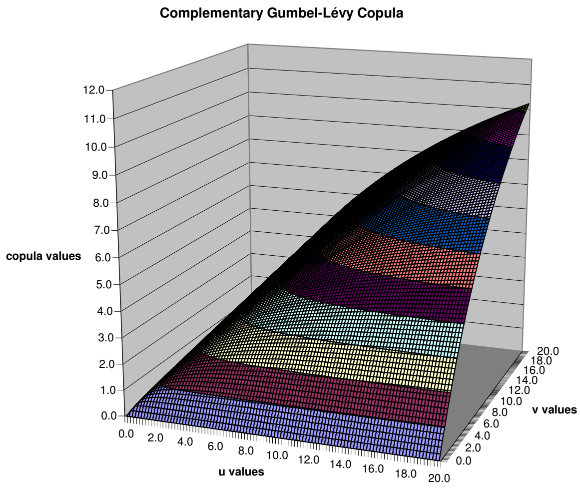

FIGURE 5. Complementary Gumbel-Lévy Copula

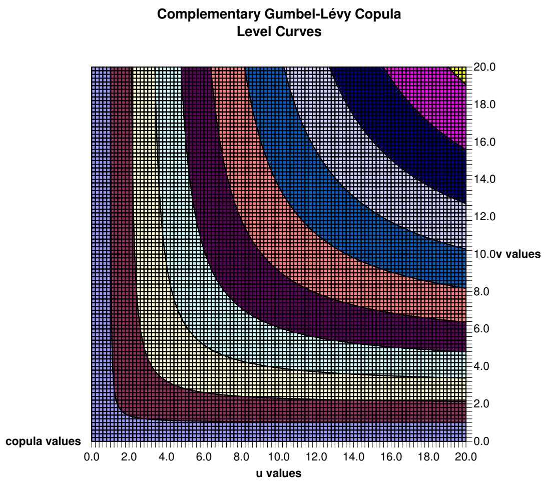

FIGURE 6. Complementary Gumbel-Lévy Copula, Level Curves

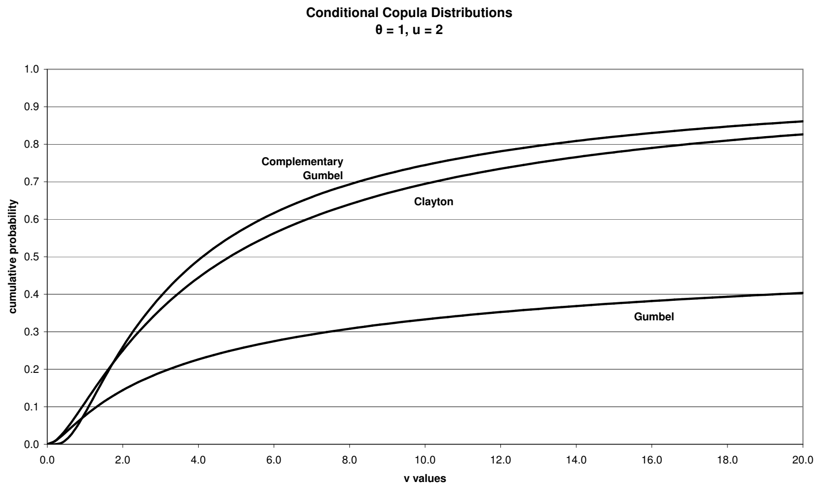

FIGURE 7. Conditional Copula Distributions

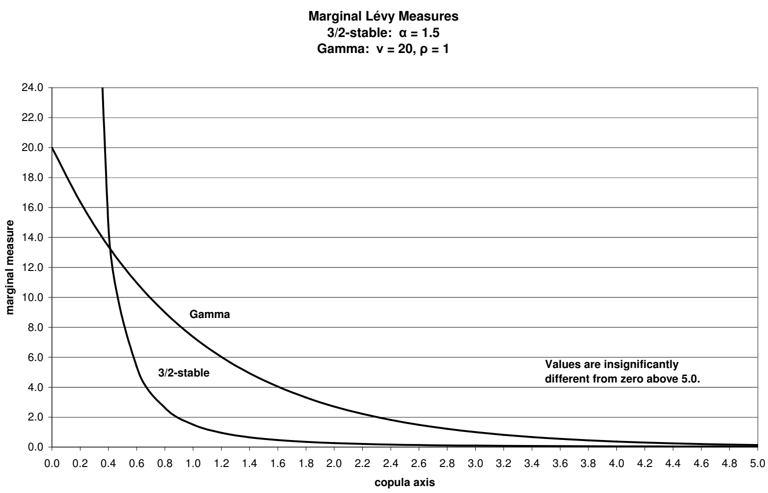

FIGURE 8. Marginal Lévy Measures

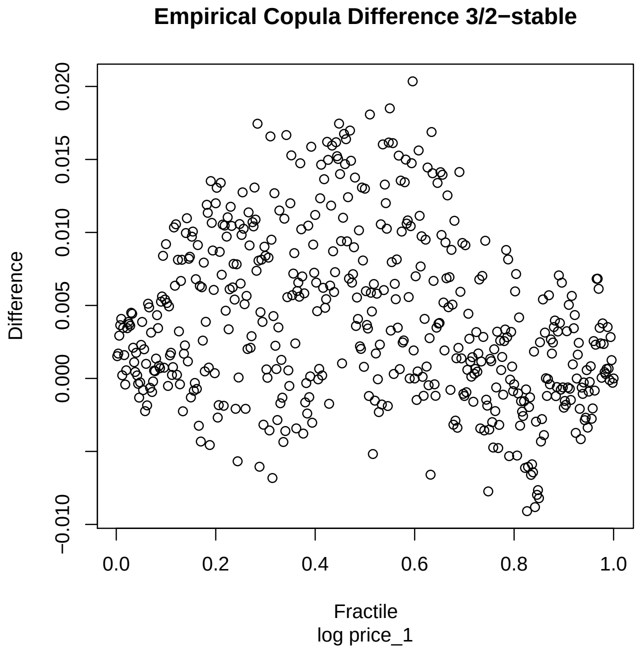

FIGURE 9. Clayton: Copula, log price 1, 3/2-stable

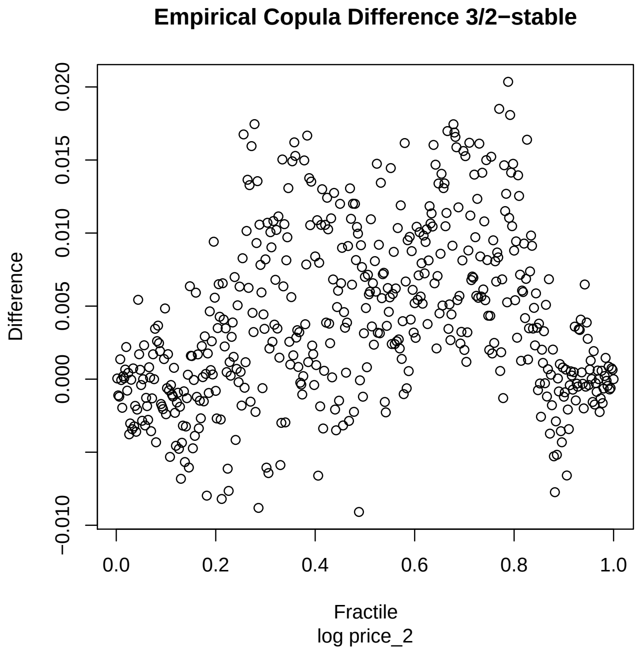

FIGURE 10. Clayton: Copula, log price 2, 3/2-stable

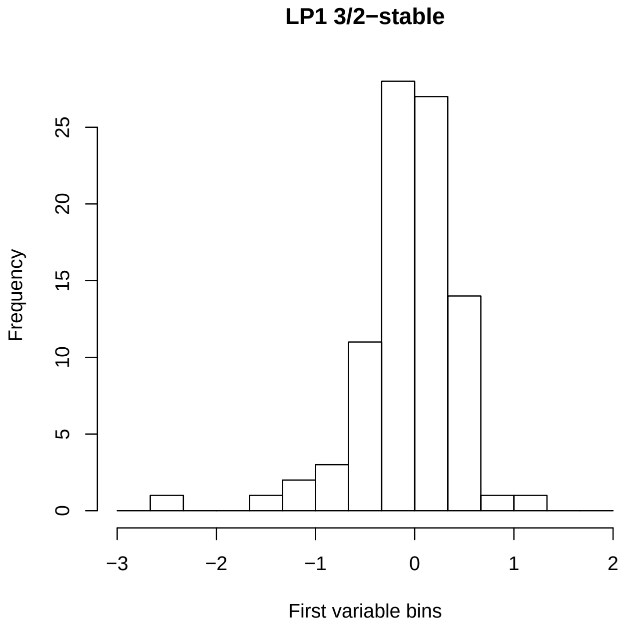

FIGURE 11. Clayton: Histogram, log relative 1, 3/2-stable

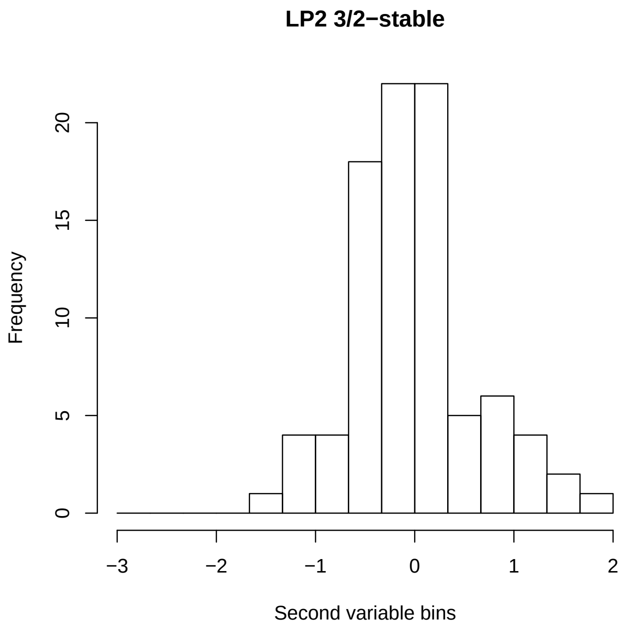

FIGURE 12. Clayton: Histogram, log relative 2, 3/2-stable

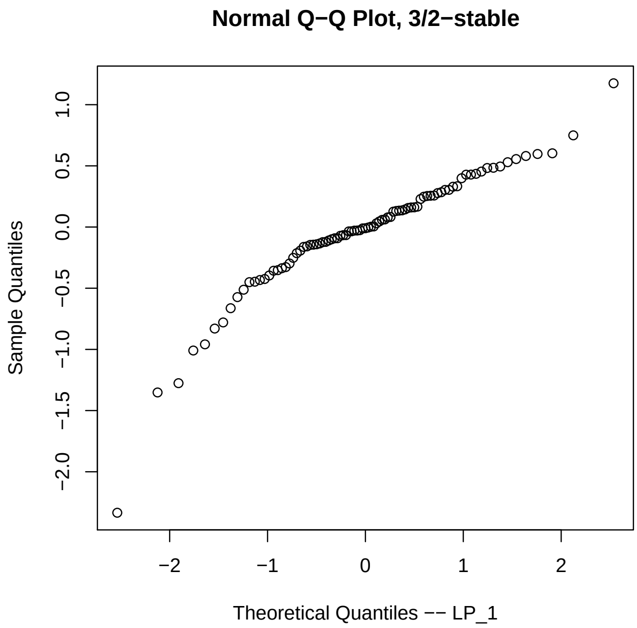

FIGURE 13. Clayton: QQ – normal plot, log relative 1, 3/2-stable

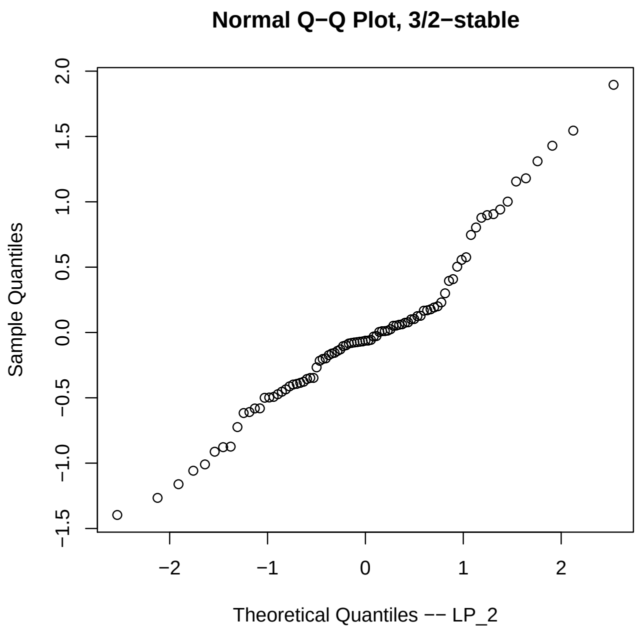

FIGURE 14. Clayton: QQ – normal plot, log relative 2, 3/2-stable

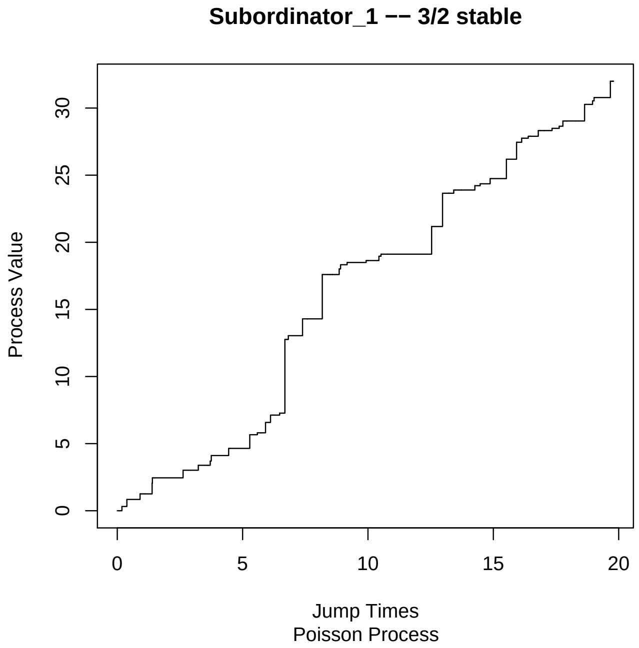

FIGURE 15. Clayton: Subordinator, variable 1, 3/2-stable

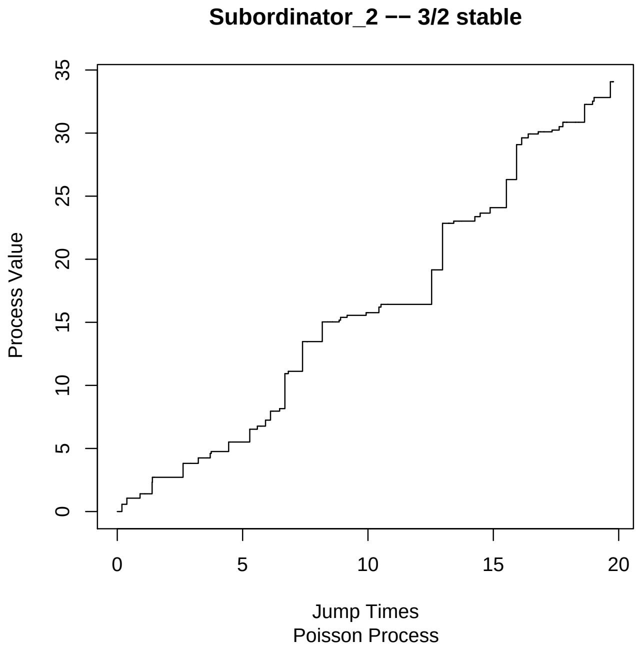

FIGURE 16. Clayton: Subordinator, variable 2, 3/2-stable

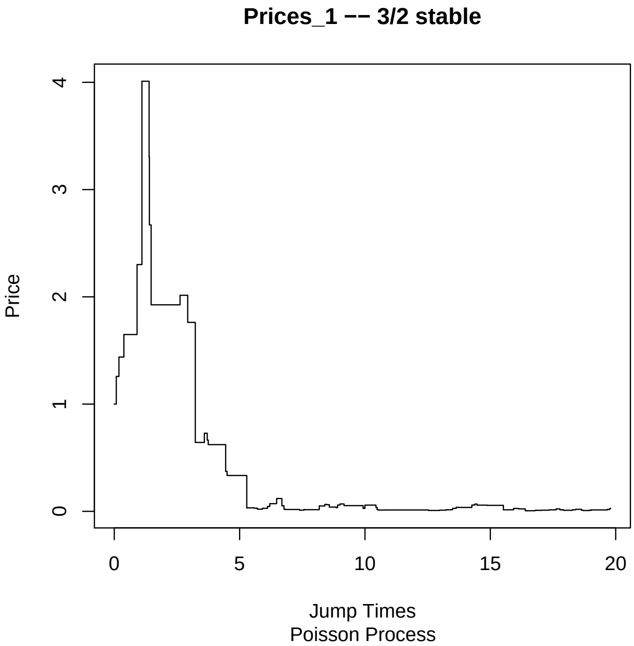

FIGURE 17. Clayton: Prices, variable 1, 3/2-stable

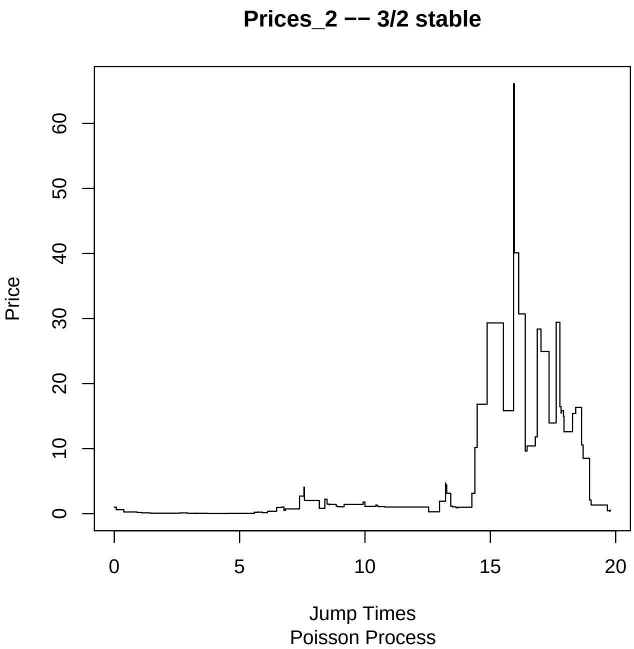

FIGURE 18. Clayton: Prices, variable 2, 3/2-stable

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